WALLIS' PROOF OF $\sqrt{-1}$ AS GEOMETRIC MEAN

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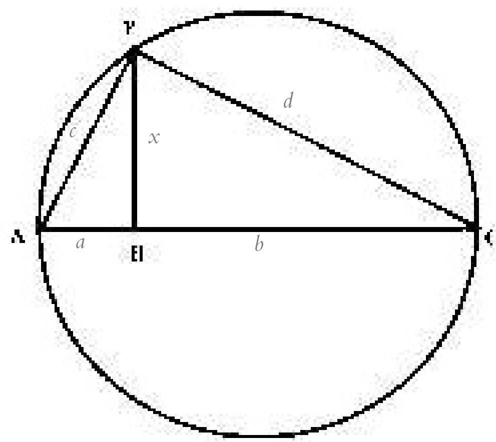

There are three right angled triangles here: ABP, APC, and BPC.

So Pythagoras' Theorem gives us

$$\begin{aligned}
&(1) & a^2 + x^2 &= c^2 \\
&(2) & b^2 + x^2 &= d^2 \\
&(3) & c^2 + d^2 &= (a+b)^2
\end{aligned}$$

Add (1) and (2) together and putting it with the expansion of (3) gives

$$a^2 + b^2 + 2x^2 = a^2 + b^2 + 2ab \tag{4}$$

Cancel out common terms gives $x^2 = ab$, so $x = \sqrt{ab}$, which is the formula for the geometric mean: $\frac{a}{x} = \frac{x}{b}$.

Note that this still holds when $a = -1$ and $b = 1$, so $\sqrt{-1}$ is the geometric mean between -1 and 1.

The geometric mean is also called the *mean proportional*. This is how the mathematicians of the 19th Century, such as Gauss, understood $\sqrt{-1}$.

Del Ferro's Formula for finding roots of the "depressed cubic": $x^3 = bx + c$.

$$\sqrt[3]{\frac{c}{2}+\sqrt{\frac{c^2}{4}-\frac{b^3}{27}}}+\sqrt[3]{\frac{c}{2}-\sqrt{\frac{c^2}{4}-\frac{b^3}{27}}}$$

Clearly when $\frac{b^3}{27} > \frac{c^2}{4}$ the sqare root will be the square root of a negative number. And yet the roots of the cubic may may still be real. It was this that first convinced mathematicians in the 16th Century that $\sqrt{-1}$ should be taken seriously. Raphael Bombelli (1526–72) seems to have been the first do so—in his *L'Algebra*.

Of course quadratic equations can also be solved, by a formula that everyone learns in high school.

Quartics (4th powers) can also be solved.

But there is no possible formula for solving an equation of the 5th power—a quintic. This is a deep fact proven by the Norwegian mathematician Niels Henrik Abel in the 19th Century (1823)—at the age of 21.

Gauss thought the term "imaginary", or "impossible", for $\sqrt{-1}$ was highly misleading, in the light of its role as mean proportional between -1 and 1. He suggested instead "direct", "inverse", and "lateral" unity for 1, -1, and $\sqrt{-1}$.

It was through Gauss, early in the 19th Century, that complex numbers and $\sqrt{-1}$ became generally accepted.

But of course there are holdouts even today who refuse to believe in them—and every high school student reenacts the same rebellion, whenever he or she hears of them. Because they have been told from the very beginning that -1 can have no square root!

But it does: it's just that it can't be a real number, but is rather $\sqrt{-1}$. Which we denote as i.

Meta-Principle of Reductive Analysis

If there is a means of coming to learn about X—a means by which we master the concept of X—then there is a reductive analysis which simply identifies X with that means.

Examples: